Capillarity Vol. 12, No. 3, p. 80-88, 2024

Invited review

Solitonic connections in capillarity theory: A review

Colin Rogers*

School of Mathematics and Statistics, The University of New South Wales, Sydney NSW 2052, Australia

Keywords:
Solitonic
capillarity
connections

Cited as:
Rogers, C. Solitonic connections in capillarity theory: A review. Capillarity, 2024, 12(3): 80-88.
https://doi.org/10.46690/capi.2024.09.03

Abstract:
A review is presented here of research to date on the application of model parameter-dependent constitutive laws for which capillarity systems admit underlying solitonic structure with their characteristic key properties such as invariance under Bäcklund transformations and admittance of Painlevé reduction. The classical Korteweg capillarity system and its extensions are considered. Reductions to the canonical solitonic nonlinear Schrödinger and its resonant nonlinear Schrodinger equation extension containing a de Broglie-Bohm potential are exhibited in turn for certain model constitutive relations. A capillarity analogue of the classical Kármán-Tsien model law of gasdynamics is shown to have a key role in such canonical reductions. A novel geometric link between a Korteweg capillarity system and the classical Da Rios system of hydrodynamics is recorded. Invariance of capillarity systems under multi-parameter Bäcklund transformations is detailed and applied. Gausson and q-gaussion phenomena in certain capillarity systems is described with concomitant classes of exact solutions. A Lagrangian encapsulation of a Korteweg capillarity system is presented whereby reduction is made to the canonical Boussinesq equation.

1. Introduction

Here, a review is to be presented of model constitutive relations in capillarity theory which allow reduction to certain canonical equations of modern soliton theory. The characteristic key properties of the latter such as being amenable to the inverse scattering transform and admittance of invariance under Bäcklund transformations whereby multi-soliton exact solutions may iteratively generated are accordingly inherited by the capillarity systems.

In nonlinear continuum mechanics the application of model multi-parameter constitutive laws for which the governing equilibrium or dynamical equations become analytically tractable is well-established. In gasdynamics, this approach originated in work on gas jets (Chaplygin, 1904). Thus, model (p,ρ) pressure-pressure density relations of the type:

$$p=\mathbb{A}+\frac{\mathbb{B}}{\rho}, \quad (\mathbb{A},\ \mathbb{B}\ \in\ \mathbb{R}) \tag{1}$$

as introduced therein have been subsequently applied to model real gas response in both subsonic and supersonic gasdynamics (Tsien, 1939; Von Kármán, 1941; Coburn, 1945). It will be seen that analogous such Kármán-Tsien relations prove key in the reduction of certain model capillarity systems to tractable solitonic canonical form. Matrix Bäcklund transformations were subsequently constructed in a systematic manner which reduce the classical hodograph system of gasdynamics to appropriate canonical forms in subsonic, transonic and supersonic flow régimes (Loewner, 1950). This was established for certain privileged multi-parameter gas laws which were then applied to approximate real gas behaviour. This work and its physical applications have been described in detail in a monograph on Bäcklund transformations (Rogers and Shadwick, 1982).

Model constitutive laws in subsonic gasdynamics were subsequently constructed via the action on the hodograph system of a novel class of infinitesimal Bäcklund transformations (Loewner, 1952). It was later to be established that, remarkably, this class on appropriate re-interpretation and extension has wide application in modern soliton theory (Rogers, 2022a). Thus, a matrix linear representation was thereby constructed

*Corresponding author.
E-mail address: c.rogers@unsw.edu.au (C. Rogers).

Yandy Scientific Press

Received May 14, 2024; revised June 4, 2024; accepted June 21, 2024; available online June 25, 2024.

which is associated with a novel master 2+1-dimensional solitonic system (Konopelchenko, 1992, 1993). Reductions of the latter include 2+1-dimensional integrable versions of the principal chiral fields model, the Toda lattice scheme, and, notably a new symmetric 2+1-dimensional solitonic sine-Gordon system. The latter, like the previously established canonical 2+1-dimensional solitonic extensions of the nonlinear Schrödinger and Korteweg-de Vries equations as embodied in the Davey-Stewartson and Nizhnik-Veselov-Novikov equations respectively, is symmetric in the spatial variables contained therein. An auto-Bäcklund transformation for the latter was subsequently derived (Konopelchenko et al., 1992) and its admittance of the Painlevé properly established in a systematic Lie group analysis (Clarkson et al., 1996).

In nonlinear elastostatics for certain model deformation constitutive laws, Bäcklund transformations of Loewner-type have been applied to a classical elastostatic system due to Neuber (1958) descriptive of the stress distribution in shear-strained isotropic elastic bodies (Clements and Rogers, 1975). The stress and warping distributions were thereby obtained for a class of boundary indentation problems, notably for Neuber-Sokolovsky model stress-deformation constitutive relations. A key connection between these results and a canonical soliton system was subsequently established and applied to construct an auto-Bäcklund transformation together with an associated nonlinear superposition principle for the Neuber elastostatics system (Rogers and Schief, 2010a).

In nonlinear elastodynamics, model stress-strain laws were introduced by Cekirge and Varley (1973) in an extensive analysis of the uniaxial transmission and reflexion of pulses in bounded elastic regions. These laws may be embedded in a multi-parameter class of stress-strain laws as generated via Bäcklund transformations of Loewner-type applied in a 1+1-dimensional elastodynamic context (Konopelchenko and Rogers, 1993). Infinitesimal Bäcklund transformations were subsequently applied to this Lagrangian encapsulation of the nonlinear elastodynamic system to derive model stress-strain laws associated with reduction to the canonical sinh-Gordon soliton equation (Rogers et al. , 2007b). Importantly, these models characteristically exhibit an interior change of concavity. This type of material response is encountered in the compression of polycrystaline materials and in nickel-titanium as used extensively in shape memory alloys. A Bäcklund transformation admitted by the sinh-Gordon reduction at the level of the stress-strain laws was subsequently derived (Rogers and Schief, 2010b). The single action on the classical Hooke's law generates the model stress-strain laws of Cekirge and Varley (1973). Iterated action of the Bäcklund transformation imbeds the latter in a wide multi-parameter class of model constitutive laws associated with sinh-Gordon solitonic canonical reduction. The nonlinear superposition principle associated with the Bäcklund transformation acting on the (T,e)-stress strain laws turns out to be nothing but the classical permutability theorem for the solitonic potential Korteweg-de Vries hierarchy (Rogers and Schief, 2002).

The preceding attests to the diverse physical applications of model constitutive laws and their key role in the reduction to tractable canonical form of certain classical governing systems in gasdynamics, nonlinear elastostatics and elastodynamics. Here, a review is to be presented of the research to date on application of model constitutive relations for which capillarity systems admit underlying solitonic structure with its characteristic integrability properties.

2. An extended Korteweg capillarity system

The classical Korteweg capillarity system:

$$\begin{gathered}\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho\ \mathbf{q})=0\\ \frac{\partial \mathbf{q}}{\partial t}+\mathbf{q}\cdot\nabla\mathbf{q}-\nabla\left[\kappa(\rho)\nabla^2\rho+\frac{1}{2}\kappa'(\rho)|\nabla\rho|^2+\Pi(\rho)\right]=\mathbf{0}\end{gathered} \tag{2}$$

has been the subject of an extensive literature since its introduction (Nimmo, 1992). Importantly, in terms of solitonic connections, it has been subsequently established that the system (Eq. (2)) may be set in the context of a more general isothermal, inviscid capillarity system, namely (Antanovskii, 1996):

$$\begin{gathered}\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho\ \mathbf{q})=0\\ \frac{\partial \mathbf{q}}{\partial t}+\mathbf{q}\cdot\nabla\mathbf{q}+\nabla\left[\frac{\delta(\rho\ \mathscr{E})}{\delta\rho}-\Pi\right]=\mathbf{0}\end{gathered} \tag{3}$$

wherein ρ is the density of the capillarity liquid, $\mathbf{q}$ is the velocity in its motion and $\mathscr{E}(\rho,\alpha)$ with $\alpha=|\nabla\rho|^2/2$ is the specific free energy. Here:

$$\frac{\delta\Phi}{\delta\rho}:=\frac{\partial\Phi}{\partial\rho}-\nabla\left(\frac{\partial\Phi}{\partial\alpha}\nabla\rho\right) \tag{4}$$

and Π is an external potential. If due to gravity, the latter is determined by the relation $\Pi=-\rho\ g$. The quantity:

$$\zeta=\frac{\delta}{\delta\rho}(\rho\mathscr{E}) \tag{5}$$

is termed the chemical potential of this capillarity system. The classical Korteweg system (Eq. (2)) is retrieved as the specialisation:

$$\mathscr{E}(\alpha,\rho)=\kappa(\rho)\frac{\alpha}{\rho}+\frac{\lambda}{\rho},\quad (\lambda\ \in\ \mathbb{R}) \tag{6}$$

in the system (Eq. (3)).

3. Canonical nonlinear Schrödinger reduction

Nonlinear Schrödinger (NLS) reduction of the capillarity system (Eq. (3)) in the irrotational case and with $\Pi=0$ was subsequently derived for a class of model energy relations $\mathscr{E}(\rho,\alpha)$ (Antanovskii et al., 1997). Thus, with $\mathbf{q}=\nabla\phi$ where ϕ is the velocity potential, the capillarity system (Eq. (2)) then consists of the continuity equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\nabla\phi)=0 \tag{7}$$

together with the Bernoulli integral:

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+\frac{\delta(\rho\mathscr{E})}{\delta\rho}=\mathbb{B}(t) \tag{8}$$

admitted by the momentum equation. Herein, the arbitrary $\mathbb{B}(t)$ may be absorbed in the potential ϕ and is accordingly set zero.

On introduction of the classical Madelung representation:

$$\Psi = \rho^{1/2} e^{i\phi/2} \tag{9}$$

the capillarity system may be encapsulated in the generalised NLS-type equation:

$$i\,\frac{\partial\Psi}{\partial t} + \nabla^2\Psi + \left[-\frac{\nabla^2|\Psi|}{|\Psi|} - \frac{1}{2}\frac{\delta(\rho\mathscr{E})}{\delta\rho} + \frac{\Pi}{2}\right]\Psi = 0 \tag{10}$$

incorporating a de Broglie-Bohm potential term $\nabla^2|\Psi|/|\Psi|$. In Antanovskii et al. (1997) it was established that with model energy laws of the type:

$$\mathscr{E}(\rho,\alpha) = \frac{1}{2}\frac{|\nabla\rho|^2}{\rho^2} - \frac{\mu}{\rho} - \nu\rho = \alpha - \frac{\mu}{\rho} - \nu\rho, \quad (\mu, \nu \in \mathbb{R}) \tag{11}$$

with $\Pi = 0$, reduction of Eq. (10) is made to the nonlinear Schrödinger equation:

$$i\,\frac{\partial\Psi}{\partial t} + \nabla^2\Psi + \nu|\Psi|^2\Psi = 0 \tag{12}$$

In 1+1-dimensions accordingly, the canonical solitonic NLS equation:

$$i\,\frac{\partial\Psi}{\partial t} + \frac{\partial^2\Psi}{\partial x^2} + \nu|\Psi|^2\Psi = 0 \tag{13}$$

results. The corresponding encapsulated capillarity system then inherits the characteristic key properties of Eq. (13) such as admittance of the inverse scattering procedure (Ablowitz and Clarkson, 1991) and invariance under a Bäcklund transformation whereby multi-soliton solutions may be iteratively generated via action of an associated nonlinear superposition principle (Rogers and Schief, 1998; and literature cited therein).

Model parameter-dependent energy relations $\mathscr{E}(\rho,\alpha)$ were subsequently constructed which result in resonant NLS equations incorporating a de Broglie-Bohm potential (Rogers and Schief, 1999). In this connection, with:

$$\mathscr{E}(\rho,\alpha) = \frac{\lambda|\nabla\rho|^2}{2\rho^2} - \frac{\mu}{\rho} - \nu\rho, \quad (\lambda,\ \mu,\ \nu \in \mathbb{R}) \tag{14}$$

reduction of Eq. (10) is made to what is termed the resonant NLS equation:

$$i\,\frac{\partial\Psi}{\partial t} + \nabla^2\Psi + \left[(\lambda-1)\frac{\nabla^2|\Psi|}{|\Psi|} + \nu|\Psi|^2\right]\Psi = 0 \tag{15}$$

If $\lambda < 0$, the latter can admit novel solitonic fission or fusion phenomena (Pashaev et al., 2008). This case anses notably in cold plasma physics (Lee et al., 2007; Rogers and Clarkson, 2018). If $\lambda > 0$ then, importantly, the de Broglie Bohm term may be removed via an appropriate transformation (Rogers, 2014c). Accordingly, in the 1+1-dimensional case, the three-parameter class of model $\mathscr{E}(\rho,\alpha)$ laws with:

$$\mathscr{E}(\rho,\alpha) = \frac{\lambda}{2}\left(\frac{\rho_x}{\rho}\right)^2 - \frac{\mu}{\rho} - \nu\rho, \quad (\lambda > 0) \tag{16}$$

result in encapsulation of the capillarity system in the canonical solitonic nonlinear Schrödinger equation.

In terms of tractable symmetry reduction, such has been obtained to a canonical Ermakov-Painlevé II equation for Korteweg capillarity systems encapsulated in an extension of the resonant NLS equation involving the addition of a triad of terms in the amplitude $|\Psi|$ (Rogers and Clarkson, 2017). The Ermakov-Painlevé II equation is linked via the relation $|\Psi| = \rho^{1/2}$ to the classical $\mathrm{P_{XXXIV}}$ equation. The latter arises notably in the analysis of certain boundary value problems for the Nernst-Planck electrolytic system where it determines ion concentration distributions (Bass et al., 2010). In a capillarity context, a resonant NLS encapsulation was applied in the isolation of a Ermakov-Painlevé II symmetry reduction valid for a multi-parameter class of free energy functions. Iterated application of an admitted Bäcklund transformation was used to generate novel classes of exact solutions of the nonlinear capillarity system in terms of Yablonskii polynomials or classical Airy functions. A $\mathrm{P_{XXXIV}}$ equation derived for the density in the capillarity system was shown to correspond to the symmetry reduction of its Bernoulli integral of motion.

It is remarked that coupled Ermakov-Painlevé II systems were originally derived as symmetry reductions of a multi-component resonant system of Manakov-type (Rogers, 2014a). In terms of physical applications, in nonlinear elastodynamics, Ermakov-Painlevé II reduction was obtained in connection with stress evolution in a class of generalised hyperelastic Mooney-Rivlin type materials under a class of shearing motions.

4. A Kármán-Tsien type constitutive relation in capillarity theory

Under the Madelung transformation (Eq. (9)), the classical Korteweg capillarity system with specific energy relation (Eq. (6)) becomes:

$$\begin{aligned} &i\,\frac{\partial\Psi}{\partial t} + \nabla^2\Psi \\ &+ \left[-\frac{\nabla^2|\Psi|}{|\Psi|} + \frac{1}{2}\kappa(\rho)\nabla^2\rho + \frac{1}{2}\kappa'(\rho)(\nabla\rho)^2 + \frac{\Pi}{2}\right]\Psi = 0 \end{aligned} \tag{17}$$

This NLS equation is not integrable in general. However, it has been established (Rogers, 2014b) that, under a plane wave packet ansatz, it admits an integrable Hamiltonian reduction with a Kármán-Tsien type constitutive relation:

$$\kappa(\rho) = \mathbb{A} + \frac{\mathbb{B}}{\rho}, \quad (\mathbb{A},\ \mathbb{B} \in \mathbb{R}) \tag{18}$$

adopted in a capillarity context. An elliptic vortex ansatz introduced therein leads to a reduction encapsulated in an integrable Ermakov-Ray-Reid subsystem with underlying Hamiltonian structure. Nonlinear coupled systems of this kind arise in 2+1-dimensional rotating shallow water theory (Rogers and An, 2010). Ermakov-Ray-Reid systems likewise are likewise important in nonlinear optics (Rogers et al., 2010, 2012; and literature cited therein). In that setting, they can, in particular, model the size and shape evolution of a light spot and wave front in an elliptical Gaussian beam (Cornolti et al., 1990; Goncharenko et al., 1991). In magneto-gasdynamics, Ermakov-Ray-Reid systems occur in the description of spinning gas cloud evolution (Rogers and An, 2012).

4.1 A geometric integrable connection: The classical da rios system

The geometric connection between important aspects of modern soliton theory and certain motions of inextensible curves has its genesis in the analysis by Scott (1975) of the spatial motion of an isolated vortex filament in an unbounded inviscid liquid. Therein, what is now termed the Da Rios system was derived, namely (Ricca, 1991; Rogers and Schief, 2002):

$$\begin{aligned} \kappa_t^* &= -2\kappa_s^* - \kappa^* \tau_s^* \\ \tau_t^* &= \left(-\tau^{*2} + \frac{\kappa_{ss}^*}{\kappa^*} + \frac{1}{2}\kappa^{*2}\right)_s \end{aligned} \tag{19}$$

where κ^*, τ^* denote, in turn, the curvature and torsion of the vortex filament while s is a measure of arc length. It was established in Rogers and Schief (2014b) that remarkably, with:

$$\kappa^* = \rho^{1/2}, \quad \tau^* = \frac{q}{2} \tag{20}$$

and $s \to x$, a 1+1-dimensional Korteweg capillarity system results, namely:

$$\begin{aligned} \rho_t + (\rho q)_x = 0 \\ q_t + q q_x + \left(-\frac{\rho_{xx}}{\rho} + \frac{1}{2}\frac{\rho_x^2}{\rho^2} - \rho\right)_x = 0 \end{aligned} \tag{21}$$

This system is obtained for the class of model laws:

$$\mathscr{E}(\rho, \alpha) = \frac{\kappa(\rho)\alpha}{\rho} + \frac{r(\rho)}{\rho}, \quad \alpha = \frac{1}{2}\left(\frac{\rho_x}{\rho}\right)^2 \tag{22}$$

wherein:

$$\kappa(\rho) = \rho^{-1} \tag{23}$$

corresponding to a Kármán-Tsien relation and:

$$r(\rho) = r_0 + r_1\rho - \frac{\rho^2}{2}, \quad (r_0,\ r_1\ \in\ \mathbb{R}) \tag{24}$$

Under the scaling:

$$(x,t) \to \mu^{-1}(x,t), \quad \rho \to \bar{\nu}\rho, \quad (\bar{\nu} > 0, \quad \mu\ \in\ \mathbb{R}) \tag{25}$$

applied to Eq. (24), a 4-parameter class of model energy laws is obtained with:

$$\begin{gathered} \kappa(\rho) = \mu^2\rho^{-1}, \quad r(\rho) = \bar{r}_0 + \bar{r}_1\rho - \bar{\nu}\frac{\rho^2}{2} \\ \bar{r}_0,\ \bar{\mathbf{r}}_1\ \in\ \mathbb{R} \end{gathered} \tag{26}$$

It may be established that the classical Da Rios system (Eq. (19)) may be encapsulated in the 1+1-dimensional solitonic nonlinear Schrödinger equation (Hasimoto, 1972). This integrability property accordingly likewise holds for the equivalent Korteweg-type capillarity system (Eq. (21)).

4.2 Invariance under multi-parameter transformations. Application in capillarity theory

Haar (1928) in connection with a variational problem derived a class of transformations which leave invariant, up to the gas law, the governing equations of plane, steady irrotational gasdynamics. Bateman (1938) subsequently established invariance of this system under a class of what have come to be reciprocal transformations and which were there related to lift and drag phenomena. It was later to be established that these constitute particular Bäcklund transformations (Bateman, 1943). Invariant transformations in plane rotational gasdynamics were derived in a systematic manner in Rogers (1972).

Multi-parameter invariant transformations in 1+1-dimensional gasdynamics and magnetogasdynamics were constructed in Rogers (1968, 1969). Application of such a transformation was made to the analysis of the motion of a gas in the region between a pioton and a driven shock wave (Castell and Rogers, 1974).

The preceding invariant transformations characteristically involve non-trivial changes of the independent variables dependent for their validity on conservation laws admitted by the original physical system. In the present capillarity context, it was established in Rogers and Schief (2014b) that the 1+1-dimensional Antanovskii capillarity system with $\Pi = 0$, namely:

$$\begin{aligned} \rho_t + (\rho q)_x = 0 \\ q_t + q q_x + \left[\frac{\delta(\rho\mathscr{E})}{\delta\rho}\right]_x = 0 \end{aligned} \tag{27}$$

is invariant under the one parameter (χ) class of transformations:

$$\begin{gathered} \rho' = \frac{\rho}{1+\chi\rho}, \quad q' = q, \quad \alpha' = \frac{\alpha}{(1+\chi\rho)^6}, \\ \mathscr{E}'(\rho', \alpha') = \mathscr{E}(\rho, \alpha), \qquad\qquad \mathbb{M} \\ dx' = (1+\chi\rho)dx - \chi\rho\ dt, \quad t' = t, \\ 0 < |1+\chi\rho| < \infty \end{gathered} \tag{28}$$

This corresponds in the gasdynamic reduction to an invariant transformation due to Movsesian (1967). This has physical application in the analysis of 1+1-dimensional piston-driven motions. In Rogers and Malomed (2018) a Madelung nonlinear optics system was shown to be invariant under a novel class of two-parameter Movsesian-type transformations. Model optical laws of Kármán-Tsien type were thereby derived.

In the capillarity context, the Movsesian type transformation was applied in Rogers and Schief (2014b) to a seed class of travelling wave packet representations in the 1+1-dimensional version of the resonant NLS encapsulation (Eq. (15)). This generated a novel class of exact solutions of the associated capillarity system in terms of the classical Weierstrass elliptic function.

A 1+1-dimensional resonant NLS encapsulation of a capillarity system in which an external gravitational potential $\Pi = -\rho\ g$ is present was introduced in Rogers (2016), namely:

$$i\,\Psi_t + \Psi_{xx} + \left[-(1-\mathbb{C})\frac{|\Psi|_{xx}}{|\Psi|} - \frac{g}{2}|\Psi|^2 - \frac{1}{2}\ r'(\rho)\right]\Psi = 0 \tag{29}$$

with $\mathscr{E}(\rho, \alpha)$ of the type (Eq. (22)) and Kármán-Tsien type relation:

$$\kappa(\rho)=\frac{\mathbb{C}}{\rho}, \quad (\mathbb{C}>0) \tag{30}$$

together with:

$$r'(\rho)=\alpha\rho+\beta\rho^2, \quad (\alpha,\ \beta\ \in\ \mathbb{R}) \tag{31}$$

A gauge transformation was applied to obtain reduction of (Eq. (29)) to a resonant quintic derivative NLS equation. A wave packet ansatz resulted in a reduction whose integrals of motion allowed the construction of two classes of exact solution generated via an algorithmic procedure in terms of *dn* and *cn* Jacobi elliptic functions. These, in turn, produce novel such solutions of the capillarity systems associated with the NLS encapsulation (Eq. (29)).

A direct corallary of the invariance $\mathbb{M}$ embodied in the relations of (Eq. (28)) is that the classical Korteweg capillarity system:

$$\begin{gathered}\rho_t+(\rho q)_x=0\\ q_t+qq_x+\left[-\kappa(\rho)\rho_{xx}-\frac{1}{2}\ \kappa'(\rho)\rho_x^2+r'(\rho)\right]_x=0\end{gathered} \tag{32}$$

is invariant under the class of transformations:

$$\begin{gathered}\rho^*=\frac{\rho}{1+\chi\rho}, \quad q^*=q,\\ \kappa^*=(1+\chi)^5\kappa, \quad r^*=\frac{r}{1+\chi\rho}, \qquad \mathbb{R}^*\\ dx^*=(1+\chi\rho)dx-\chi\rho\ q\ dt, \quad t^*=t,\\ 0<|1+\chi\rho|<\infty\end{gathered} \tag{33}$$

This result is readily extended to incorporate gravitational potential $\Pi=-\rho g$ by appropriate adjustment of $r(\rho)$.

5. Gausson and q-Gaussian phenomena in capillarity theory

5.1 Gausson representations

In Da Martino et al. (2004) a capillarity system of the type:

$$\begin{gathered}\rho_t+\operatorname{div}(\rho\mathbf{q})=0\\ q_\mathbf{t}+\mathbf{q}\cdot\nabla\mathbf{q}-\nabla\left[-\mu\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}}+\nu\ln\rho\right]=\mathbf{0}\end{gathered} \tag{34}$$

was derived. This corresponds to a Kármán-Tsien $\kappa(\rho)$ relation together with:

$$r(\rho)=\nu\rho\left(-\ln\rho+1\right) \tag{35}$$

in the model energy $\mathscr{E}(\rho,\alpha)$ representation in (Eq. (27)). The system (Eq. (34)) is not integrable as it stands. However, in the irrotational case, it can be established via a Madelung-type encapsulation that it can be embodied in a logarithmic NLS equation:

$$\frac{\partial\bar{\Psi}}{\partial\bar{t}}+\nabla^2\bar{\Psi}+\bar{\nu}(\ln\bar{\Psi})\bar{\Psi}=0 \tag{36}$$

(Rogers, 2019). This turns out to admit novel exact solutions termed gaussons (Bialynicki-Birula and Mycielski, 1978) which in 1+1-dimensions admit the form:

$$\bar{\Psi}=N\exp\left[i\frac{-E\bar{t}+k(\bar{x}-k\bar{t})}{2}-\frac{\bar{\nu}(\bar{x}-k\bar{t})^2}{4}\right] \tag{37}$$

The class of invariant transformations $\mathbb{R}^*$ may be generalised and action on (Eq. (37)) applied to generate an extensive class of novel multi-parameter exact solutions to the 1+1-dimensional reduction of the capillarity system with the irrotational constraint. It is remarked that 3+1-dimensional gausson phenomena in a modulated version of (Eq. (36)) incorporating a de Broglie-Bohm potential has been recorded in a plasma physics (Rogers, 2014a). A connection was made, in particular, with experimental observations related to the expansion of laser-pulsed plasma ellipsoids into a vacuum.

5.2 q-Gaussian representations

The concepts of q-logarithmic and its inverse q-exponential function:

$$\log_q x=\frac{x^{1-q}-1}{1-q} \tag{38}$$

and:

$$\exp_q x=[1+(1-q)x]^{1/(1-q)}, \quad (x>0) \tag{39}$$

were originally introduced in a statistical mechanical context but have subsequently proved to have a broad spectrum of applications (Gell-Mann and Tsallis, 2004) and literature cited therein). In anisentropic gasdynamics (Rogers and Ruggeri, 2014) and magnetogasdynamics (Rogers and Schief, 2014a), 2+1-dimensional elliptic vortex motions have been isolated associated with q-Gaussian density distributions of the type:

$$\begin{gathered}\rho=\omega(t)\exp_q[-\mathbf{x}^TE(t)\mathbf{x}+c]\\ \mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}\end{gathered} \tag{40}$$

and with q-dependent model gas laws. In a capillarity context (Rogers, 2019) q-Gaussian exact solutions were derived for a Madelung encapsulation of a class of 1+1-dimensional Korteweg capillarity systems in a dual power law NLS equation:

$$i\ \frac{\partial\Psi}{\partial t}+\Psi_{xx}-\left[(1-\mathbb{C})\frac{|\Psi|_{xx}}{|\Psi|}+\alpha+\beta|\Psi|^{q-1}+\gamma|\Psi|^{2(q-1)}\right]\Psi=0 \tag{41}$$

incorporating a de-Broglie Bohm potential. This corresponds to model energy $\mathscr{E}(\rho,\alpha)$ laws with Kármán-Tsien type relation (Eq. (30)) together with:

$$\frac{r'(\rho)}{2}=\lambda+\beta\rho^{(q-1)/2}+\gamma\rho^{q-1} \tag{42}$$

Reduction of (Eq. (41)) was made to a dual power NLS equation:

$$i\ \frac{\partial\bar{\Psi}}{\partial\bar{t}}+\bar{\Psi}_{\bar{x}\bar{x}}+\left[\bar{\alpha}+\bar{\beta}|\bar{\Psi}|^{q-1}+\bar{\gamma}|\bar{\Psi}|^{2(q-1)}\right]\bar{\Psi}=0 \tag{43}$$

with de-Broglie Bohm term removed. The latter type of NLS equation admits q-Gaussian wave packet solutions:

$$\bar{\Psi} = N\exp\left[-E\bar{t}i + \frac{k(\bar{x}-k\bar{t})}{2}i\right]\exp_q[-a(\bar{x}-k\bar{t})^2] \tag{44}$$

with appropriate constraints on the parameters therein. In the limit $q \to 1$, Eq. (44) is consistent with the gausson-type class (Eq. (37)). Action of invariance under a class of reciprocal transformations admitted by the Korteweg capillarity system associated with the dual power NLS encapsulation (Eq. (43)) generates a two-parameter class of its exact solutions (Rogers, 2019).

6. Lagrangian representation of a korteweg capillarity system. Boussinesq solitonic reduction

Here, the Lagrangian version of the 1+1-dimensional Korteweg capillarity system (Eq. (32)) is recorded for the class of model energy relations of the type (Eq. (23)). Invariance of the system accordingly holds under the class of transformations $\mathbb{R}^*$ as set down in (Eq. (33)).

Thus, on introduction of Lagrangian variables X, T according to:

$$dX = \rho dx - \rho q dt, \quad dT = dt \tag{45}$$

so that:

$$dx = \frac{1}{\rho}\, dX + q dT$$

whence:

$$\frac{\partial}{\partial T}\left(\frac{1}{\rho}\right) - \frac{\partial q}{\partial X} = 0 \tag{46}$$

The latter constitutes the usual Lagrangian representation of the 1+1-dimensional continuity equation. The Eulerian continuity and momentum equations in the capillarity system may, in turn, be combined to produce the conservation law:

$$\frac{\partial}{\partial t}(\rho q) + \frac{\partial}{\partial x}\left[\Sigma + \rho q^2 - \kappa(\rho)\left(\rho_{xx}\rho - \frac{\rho_x^2}{2}\right) - \frac{\rho}{2}\kappa'(\rho)\rho_x^2\right] = 0 \tag{47}$$

wherein:

$$\Sigma(\rho) = \rho r'(\rho) - r(\rho) \tag{48}$$

In the present Lagrangian representation (Eq. (47)) becomes:

$$\frac{\partial q}{\partial T} + \frac{\partial}{\partial X}\left[\Sigma - \kappa(\rho)\left(\rho_{xx}\rho - \frac{\rho_x^2}{2}\right) - \frac{\rho}{2}\kappa'(\rho)\rho_x^2\right] = 0 \tag{49}$$

so that, on introduction of ρ' and $\mathscr{K}(\rho')$ according to (Rogers, 2022b):

$$\rho' = \frac{1}{\rho} \tag{50}$$

$$\mathscr{K}(\rho') = \frac{\kappa}{\rho'^5} \tag{51}$$

the Lagrangian conservation law:

$$\frac{\partial q}{\partial T} + \frac{\partial}{\partial X}\left[\mathscr{K}\frac{\partial^2\rho'}{\partial X^2} + \frac{1}{2}\partial\mathscr{K}/\partial\rho'\left(\frac{\partial\rho'}{\partial X}\right)^2 + \Sigma\right] = 0 \tag{52}$$

results. Elimination of q between Eqs. (46) and (52) now produces a generalised Boussinesq-type equation, namely (Rogers, 2022b):

$$\frac{\partial^2\rho'}{\partial T^2} + \frac{\partial^2}{\partial X^2}\left[\mathscr{K}(\rho')\frac{\partial^2\rho'}{\partial X^2} + \frac{1}{2}\frac{\partial\mathscr{K}}{\partial\rho'}\left(\frac{\partial\rho'}{\partial X}\right)^2 + \Sigma'(\rho')\right] = 0 \tag{53}$$

wherein:

$$\Sigma'(\rho') = \Sigma|_{\rho=\rho'^{-1}} \tag{54}$$

7. Solitonic reduction

With the specialisations:

$$\mathscr{K} = \lambda, \quad \Sigma'(\rho') = \mu\rho'^2 + \nu\rho' \tag{55}$$

corresponding to the 4-parameter model energy law $\mathscr{E}(\rho, \alpha)$ with:

$$\kappa = \frac{\lambda}{\rho^5}, \quad r(\rho) = -\frac{\mu}{3}\frac{1}{\rho^2} + \xi\rho - \nu \qquad (\lambda,\ \mu,\ \nu,\ \xi\ \in\ \mathbb{R}) \tag{56}$$

in the Korteweg capillarity system (Eq. (32)), reduction is made to the classical integrable solitonic Boussinesq equation:

$$\rho'_{TT} + \lambda\rho'_{XXXX} + \mu(\rho'^2)_{XX} + \nu\rho'_{XX} = 0 \tag{57}$$

It was Boussinesq (1872) who originally introduced such a nonlinear equation in a mathematical analysis of the evolution of long waves in shallow water. It constitutes a seminal solitonic equation and as such is amenable to the inverse scattering transform. In addition, it admits an auto-Bäcklund transformation (Hirota and Satsuma, 1977; Rasin and Schiff, 2017). It has a range of physical applications, notably in nonlinear lattice theory (Toda, 1975). In Clarkson and Winternitz (1999), an account of the classical and non-classical Lie group symmetry reductions of the solitonic Boussinesq equation is presented.

8. A 2+1-dimensional capillarity model system

The resonant solitonic Davey-Stewartson system as introduced in Rogers et al. (2009) in a 2+1-dimensional capillarity context adopts the form:

$$\begin{gathered} i\frac{\partial\Psi}{\partial t} + \nabla^2\Psi + (\delta-1)\frac{\nabla^2|\Psi|}{|\Psi|}\Psi + \gamma|\Psi|^2\Psi + \frac{\Pi\Psi}{2} = 0 \\ \Pi_{xx} - \Pi_{yy} + 4\gamma(|\Psi|^2)_{xx} = 0 \\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{gathered} \tag{58}$$

The decomposition $\Psi = e^{R-iS}$ together with the relations:

$$\rho = e^{2R}, \quad \mathbf{q} = -2\nabla S \tag{59}$$

leads to a Madelung-type system:

$$\begin{gathered} \rho_t + \nabla(\rho\mathbf{q}) = 0 \\ \frac{\partial\mathbf{q}}{\partial\mathbf{t}} + \mathbf{q}\cdot\nabla\mathbf{q} - \nabla\left[2\delta\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}} + 2\gamma\rho + \Pi\right] = 0 \end{gathered} \tag{60}$$

where:

$$\Pi_{xx} - \Pi_{yy} + 4\gamma\rho_{xx} = 0 \tag{61}$$

In Rogers et al. (2009), a 2+1-dimensional capillarity system (Eq. (2)) was shown to be reducible to a resonant Davey-Stewartson system for a 3-parameter class of model free energy expressions $\mathscr{E}(\rho,\alpha)$. Two-soliton interaction in this nonlinear system was thereby analysed via a bilinear operator representation of a type originally introduced by Hirota (1971) in connection with solitonic collision phenomena admitted by the canonical Korteweg-de Vries equation.

In Rogers and Pashaev (2011), the 2+1-dimensional resonant Davey-Stewartson system (Eq. (58)) was shown to be equivalent to a novel 2+1-dimensional extension of the Kaup-Broer hydrodynamic system. The latter was thereby seen via a bilinear operator representation to exhibit resonant soliton interaction. It is remarked that application of the $\bar{\partial}$-dressing method (Konopelchenko, 1992) to 2+1-dimensional integrable extension of the Kaup-Bauer system has been the subject of a recent extensive work in Nabelek and Zakharov (2020).

In Rogers et al. (2007a), a 2+1-dimensional capillarity system encapsulated in the canonical solitonic Davey-Stewartson system was shown to admit novel double periodic type wave patterns for a class of two-parameter model $\mathscr{E}(\rho,\alpha)$ laws together with an external driving mechanism embodied in Π.

9. Conclusion

It remains, in particular, to undertake the systematic investigation of capillarity systems under Lie group invariance in order to isolate Painlevé symmetry reduction indicative of underlying solitonic structure. Thus, such an investigation is suggested for a novel generalised Boussinesq capillarity model as recently derived via a Lagrangian representation.

In nonlinear continuum mechanics, model constitutive laws have been extensively applied to approximate physical relations derived empirically. The practical application in capillarity theory of parameter-dependent model constitutive laws as derived via Lie group and reciprocal invariance has yet to be undertaken in general.

Conflict of interest

The author declares no competing interest.

References

Ablowitz, M. J., Clarkson, P. A. Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge, UK, Cambridge University Press, 1991.

Antanovskii, L. K. Microscale theory of surface tension. Physical Review E, 1996, 54: 6285-6290.

Antanovskii, L. K., Rogers, C., Schief, W. K. A note on a capillarity model and the nonlinear Schrödinger equation. Journal of Physics A: Mathematical and General, 1997, 30: L555-L557.

Bass, L., Nimmo, J. J. C., Rogers, C., et al. Electrical structures of interfaces: A Painlevé II model. Proceedings of the Royal Society A, 2010, 466: 2117-2130.

Bateman, H. The lift and drag functions for an elastic fluid in two-dimensional irrotational flow. Proceedings of the National Academy of Sciences USA, 1938, 24: 246-251.

Bateman, H. The transformation of partial differential equations. Quarterly of Applied Mathematics, 1943, 1: 281-295.

Bialynicki-Birula, I., Mycielski, J. Gaussons: Solutions of the logarithmic Schrödinger equation. Physica Scripta, 1978, 20: 530-544.

Boussinesq, J. Théorie des ondos et des remous qui propagent lelong d'un canal rectangulaire horizontal en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. Journal de Mathématiques Pures et Appliquées, 1872, 17: 55-108.

Castell, S. P., Rogers, C. Application of invariant transformations in one-dimensional non-steady gasdynamics. Quarterly of Applied Mathematics, 1974, 32: 241-251.

Cekirge, H. M., Varley, E. Large amplitude waves in bounded media I: Reflexion and transmission of large amplitisds shockless pulses at an interface. Proceedings of the Royal Society A, 1973, 273: 261-313.

Chaplygin, S. A. On gas jets. Scientific Memoirs, Moscow University Mathematical Physics, 1904, 21: 1-121.

Clarkson, P. A., Mansfield, E. L., Milne, A. E. Symmetries and exact solutions of a (2+1)-dimensional sine-Gordon system. Philosophical Transactions of the Royal Society A, 1996, 354: 1807-1835.

Clarkson, P. A., Winternitz, P. Symmetric reduction and exact solutions of nonlinear partial differential equations, in The Painlevé Property, edited by R. Conte, Springer New York, New York, pp. 591-660, 1999.

Clements, D. L., Rogers, C. On the theory of stress concentration for shear-strained prismatical bodies with a nonlinear stress-strain law. Mathematika, 1975, 22: 34-42.

Coburn, N. The Kármán-Tsien pressure-volume relation in two-dimensional supersonic flow of compressible fluids. Quarterly of Applied Mathematics, 1945, 3: 106-116.

Cornolti, F., Luccesi, M., Zambon, B. Elliptic Gaussian beam self-focussing in nonlinear media. Optics Communications, 1990, 75: 129-135.

Da Martino, S., Falanga, M., Lauro, G., et al. Kinetic derivation of the hydrodynamic equations for capillary fluids. Physical Review E, 2004, 70: 067301.

Gell-Mann, M., Tsallis, C. Non-Extensive Entropy: Interdisciplinary Applications. Oxford, UK, Oxford University Press, 2004.

Goncharenko, A. M., Logvin, Yu. A., Samson, A. M., et al. Ermakov Hamiltonian systems in nonlinear optics of elliptic Gaussian beams. Physics Letters A, 1991, 160(2): 138-142.

Haar, A. Über adjungierte Variationsprobleme und adjungierte Extremalflachen. Mathematische Annalen, 1928, 100: 481-502.

Hasimoto, H. A soliton on a vortex filament. Journal of Fluid Mechanics, 1972, 51: 477-485.

Hirota, R. Exact solution of the Korteweg-de Vries equation for multiple collision of solitons. Physical Review Letters, 1971, 27: 1192-1194.

Hirota, R., Satsuma, J. Nonlinear evolution equations generated from the Bäcklund transformation for the Boussinesq equation. Progress in Theoretical Physics, 1977, 57: 797-807.

Konopelchenko, B. G. Introduction to Multi dimensional Integrable Equations: The Inverse Spectral Transform in 2+1-Dimensions. New York, USA, Springer New York, 1992.

Konopelchenko, B. G., Rogers, C. On (2+1) dimensional nonlinear systems of Loewner-type. Physics Letters A, 1991, 158: 391-397.

Konopelchenko, B. G., Rogers, C. On generalised Loewner systems: Novel integrable equations in 2+1-dimensions. Journal of Mathematical Physics, 1993, 22: 34-42.

Konopelchenko, B. G., Schief, W. K., Rogers, C. The 2+1-dimensional sine Gordon system: Its auto-Bäcklund transformation. Physics Letters A, 1992, 72: 39-48.

Lee, J. H., Pashaev, O. K., Rogers, C., et al. The resonant nonlinear Schrödinger equation in cold plasma physics. Application of Bäcklund-Darboux transformations and superposition principles. Journal of Plasma Physics, 2007, 73(2): 257-272.

Loewner, C. A transformation theory of partial differential equations of gasdynamics. National Advisory Committee on Aeronautics Technical Note, 1950, 2065: 1-56.

Loewner, C. Generation of systems of partial differential equations by composition of infinitesimal Bäcklund transformations. Journal d'Analyse Mathématique, 1952, 2: 219-242.

Movsesian, L. A. On an invariant transformation of equations of one-dimensional unsteady motion of an ideal compressible fluid. Prikladnaya Matematika: Mekhanika, 1967, 37: 137-141.

Nabelek, P. V., Zakharov, V. E. Solutions to the Kaup-Broer system and its (2+1)-dimensional integrable generalization via the dressing method. Physica D: Nonlinear Phenomena, 2020, 409: 132478.

Neuber, H. Kerbspannungslehre: Theorie der Spannungskonzentration Genaue Berechnung der Festigkeit. Berlin, Germany, Springer-Verlag, 1958.

Nimmo, J. J. C. A class of solutions of the Konopelchenko-Rogers equations. Physics Letters A, 1992, 168: 113-119.

Pashaev, O. K., Lee, J. H., Rogers, C. Soliton resonances in a generalized nonlinear Schrödinger equation. Journal of Physics A: Mathematical and Theoretical, 2008, 41: 452001.

Rasin, A. G., Schiff, J. Bäcklund transformations for the Boussinesq equation and merging solitons. Journal of Physics A: Mathematical and Theoretical, 2017, 50: 325202.

Ricca, R. L. Rediscovery of Da Rios equations. Nature, 1991, 352: 561-562.

Rogers, C. Reciprocal relations in non-steady one-dimensional gasdynamics. Zeitschrift für angewandte Mathematik und Physik, 1968, 19: 58-63.

Rogers, C. Invariant transformations in non-steady gasdynamics and magneto-gasdynamics. Zeitschrift für angewandte Mathematik und Physik, 1969, 20: 370-382.

Rogers, C. The construction of invariant transformations in plane rotational gasdynamics. Archives for Rational Mechanics and Analysis, 1972, 47: 36-46.

Rogers, C. A novel Ermakov-Painlevé II system. N+1-dimensional coupled NLS and elastodynamic reductions. Studies in Applied Mathematics, 2014a, 133: 214-231.

Rogers, C. Integrable substructure in a Korteweg capillarity model. A Kármán-Tsien type constitutive relation. Journal of Nonlinear Mathematical Physics, 2014b, 21(1): 74-88.

Rogers, C. Gausson-type representations in nonlinear physics: Ermakov modulation. Physica Scripta, 2014c, 80: 105208.

Rogers, C. The Korteweg capillarity system. Integrable reduction via gauge and reciprocal links. Zeitschrift für Angewandte Mathematik und Mechanik, 2016, 96: 813-823.

Rogers, C. Reciprocal gausson phenomena in a Korteweg capillarity system. Meccanica, 2019, 54: 1515-1523.

Rogers, C. On the Lagrangian version of the Korteweg capillarity system: integrability aspects. Ricerche di Matematica, 2022a, 71: 29-39.

Rogers, C. A Nonlinear Progress to Modern Soliton Theory. Cambridge, UK, Cambridge Scholars Publishing, 2022b.

Rogers, C., An, H. Ermakov-Ray-Reid systems in 2+1-dimensional rotating shallow water theory. Studies in Applied Mathematics, 2010, 125: 275-299.

Rogers, C., An, H. A non-isothermal spinning magneto-gasdynamic cloud system A Hamiltonian Ermakov integrable reduction. Noti di Matematika, 2012, 32(1): 175-191.

Rogers, C., Chow, K. W., Conte, R. On a capillarity model and the Davey-Stewartson I system: Quasi-doubly periodic wave patterns. Il Nuovo Cimento B, 2007a, 122: 105-112.

Rogers, C., Clarkson, P. A. Ermakov-Painlevé II symmetry reduction of a Korteweg capillarity system. Symmetry, Integrability and Geometry: Methods and Applications, 2017, 13: 018.

Rogers, C., Clarkson, P. A. Ermakov-Painlevé II reduction in cold plasma physics. Application of a Bäcklund transformation. Journal of Nonlinear Mathematical Physics, 2018, 25: 247-261.

Rogers, C., Malomed, B. On Madelung systems in nonlinear optics: A reciprocal invariance. Journal of Mathematical Physics, 2018, 59: 051506.

Rogers, C., Malomed, B., An, H. Ermakov-Ray-Reid reductions of variational approximations in nonlinear optics. Studies in Applied Mathematics, 2012, 129: 389-413.

Rogers, C., Malomed, B., Chow, W., et al. Ermakov-Ray-Reid systems in nonlinear optics. Journal of Physics A: Mathematical and Theoretical, 2010, 43: 455214.

Rogers, C., Pashaev, O. On a 2+1-dimensional Whitham-Broer-Kaup system: A resonant NLS connection. Studies in Applied Mathematics, 2011, 127: 141-152.

Rogers, C., Ruggeri, T. q-Gaussian integrable Hamiltonian reductions in anisentropic gasdynamics. Discrete & Continuous Dynamical Systems, 2014, 19: 2297-2312.

Rogers, C., Schief, W. K. Intrinsic geometry of the NLS equation and its auto-Bäcklund Transformation. Studies in Applied Mathematics, 1998, 101: 267-287.

Rogers, C., Schief, W. K. The resonant nonlinear Schrödinger equation via an integrable capillarity model. Il Nuovo Cimento B, 1999, 114: 1409-1412.

Rogers, C., Schief, W. K. Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory. Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002.

Rogers, C., Schief, W. K. Integrable structure in the theory of stress concentration in shear-strained nonlinear elastic materials. Studies in Applied Mathematics, 2010a, 25: 39-53.

Rogers, C., Schief, W. K. Bäcklund transformations and superposition principles in nonlinear elastodynamics. Studies in Applied Mathematics, 2010b, 124: 137-149.

Rogers, C., Schief, W. K. On q-Gaussian integrable Hamiltonican reductions in anisentropic magnetogasdynamics. Acta Applicandae Mathematicae, 2014a, 132: 515-525.

Rogers, C. Schief, W. K. The classical Korteweg capillarity system: geometry and invariant transformations. Journal of Physics A: Mathematical and Theoretical, 2014b, 47: 345201.

Rogers, C., Schief, W. K., Chow, K. W. On a novel class of model constitutive laws in nonlinear elasticity. Construction via Loewner theory. Theoretical and Mathematical Physics, 2007b, 152: 1030-1042.

Rogers, C., Shadwick, W. F. Bäcklund Transformations and Their Applications. New York, USA, Academic Press, 1982.

Rogers, C., Yip, L. P., Chow, K. W. A resonant Davey-Stewartson capillarity model system: Solitonic generation. International Journal of Nonlinear Science and Numerical Simulation, 2009, 10(3): 397-405.

Scott, A. C. The application of Bäcklund transformations to physical problems, in Bäcklund Transformations, edited by R. M. Miura, Springer-Verlag, Berlin, pp. 80-105, 1975.

Toda, M. Studies of a nonlinear lattice. Physics Reports, 1975, 8: 1-125.

Tsien, H. S. Two-dimensional subsonic flow of compressible fluids. Journal of the Aeronautical Sciences, 1939, 6: 399-407.

Von Kármán, T. Compressibility effects in aerodynamics. Journal of the Aeronautical Sciences, 1941, 8: 337-356.